§6.3 Trigonometric Equations

Objectives

2. Solve Trigonometric Equations Using a Calculator.
3. Solve Trigonometric Equations Quadratic in Form.
§6.3 Trigonometric Equations

Finding an Inverse Function vs Solving an Equation

Given $f(x) = x^2$

a) Find $f^{-1}(x)$
   
   b) Solve $f(x) = 4$

\[y = x^2\]
\[x = y^2\]
\[y = \sqrt{x}\]
\[f^{-1}(x) = \sqrt{x}\]

b) $x^2 = 4$
\[
\sqrt{x^2} = \sqrt{4}
\]
\[x = \pm 2\]
Determine whether $\theta = \pi/4$ is a solution of the equation:

$$2\sin(\theta) + \sqrt{2} = 0$$

Is $\theta = 5\pi/4$ a solution?

$$f(\theta) = 2\sin(\theta) + \sqrt{2}$$

$$f(\pi/4) = 2\sin(\pi/4) + \sqrt{2}$$

$$= 2 \cdot \frac{\sqrt{2}}{2} + \sqrt{2} = \sqrt{2} \neq 0$$

$$f(5\pi/4) = 2\sin(5\pi/4) + \sqrt{2}$$

$$= 2(\sqrt{2}/2) + \sqrt{2}$$

$$= \sqrt{2}$$
\section*{§6.3 Trigonometric Equations}

\subsection*{1. Solve Equations Involving a Single Trigonometric Function}

Determine whether $\theta = \pi/4$ is a solution of the equation:

\[ 2\sin(\theta) + \sqrt{2} = 0 \]

Is $\theta = 5\pi/4$ a solution?

Let: $f(\theta) = 2\sin(\theta) + \sqrt{2}$

\[ f\left(\frac{\pi}{4}\right) = 2\sin\left(\frac{\pi}{4}\right) + \sqrt{2} \]

\[ = 2\frac{\sqrt{2}}{2} + \sqrt{2} \]

\[ \neq 0 \]

\[ f\left(\frac{5\pi}{4}\right) = 2\sin\left(\frac{5\pi}{4}\right) + \sqrt{2} \]

\[ = 2\left(-\frac{\sqrt{2}}{2}\right) + \sqrt{2} \]

\[ = 0 \]
Solve the equation: $2\cos(\theta) - \sqrt{3} = 0$

Give a general formula for all solutions. List eight of the solutions.

$2\cos(\theta) = \sqrt{3}$

$\cos(\theta) = \frac{\sqrt{3}}{2}$

$\theta = \begin{cases} 
\frac{\pi}{6} + k2\pi, & \text{if } k \text{ is even} \\
-\frac{\pi}{6} + k2\pi, & \text{if } k \text{ is odd}
\end{cases}$

$k \in \mathbb{Z}$

$\theta = \begin{cases} 
\frac{\pi}{6}, & \frac{13\pi}{6}, & -\frac{\pi}{6}, \ldots 
\end{cases}$

$k, l, m, n$ make a list.
Solve the equation: $2 \cos(\theta) - \sqrt{3} = 0$

Give a general formula for all solutions. List eight of the solutions.

$$\theta = \pm \frac{\pi}{6} (1 + k12)$$
solve(2 \cos(\theta) - \sqrt{3} = 0, \theta)

Input interpretation:

\[
\begin{align*}
\text{solve} & \quad 2 \cos(\theta) - \sqrt{3} = 0 \\
\text{for} & \quad \theta
\end{align*}
\]

Results:

\[
\begin{align*}
\theta &= \pi \left(2n - \frac{1}{6}\right) \quad \text{and} \quad n \in \mathbb{Z} \\
\theta &= \pi \left(2n + \frac{1}{6}\right) \quad \text{and} \quad n \in \mathbb{Z}
\end{align*}
\]

\(Z\) is the set of integers
\[ \theta = \frac{\pi}{6} \left( 1 + \omega n_{1/12} \right) \]

\[ \theta = \frac{\pi}{6} \left( -1 + \omega n_{1/12} \right) \]
Solve the equation: $2 \cos(\theta) - \sqrt{3} = 0$

Give a general formula for all solutions. List eight of the solutions.

\[ 2 \cos \theta - \sqrt{3} = 0 \]
\[ 2 \cos \theta = \sqrt{3} \]
\[ \cos \theta = \frac{\sqrt{3}}{2} \]
\[ \theta = \begin{cases} 
\frac{\pi}{6} + k2\pi \\
-\frac{\pi}{6} + k2\pi 
\end{cases} , k \in \mathbb{Z} \]
\[ \theta = \pm \frac{\pi}{6} + k2\pi \]
\[ = \pm \frac{\pi}{6}(1 + k12) \]
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \(2 \cos(\theta) - \sqrt{3} = 0\)

Give a general formula for all solutions. List eight of the solutions.

\[
\theta = \left\{ \pm \frac{\pi}{6}, \pm \frac{11\pi}{6}, \pm \frac{13\pi}{6}, \pm \frac{23\pi}{6} \right\}
\]
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \(2 \sin(2\theta) - \sqrt{2} = 0, \quad 0 \leq \theta < 2\pi\)

\[
2 \sin(2\theta) = \sqrt{2} \\
\sin(2\theta) = \frac{\sqrt{2}}{2} \\
2\theta = \left\{ \begin{array}{l}
\frac{\pi}{4} + k2\pi \\
\frac{3\pi}{4} + k2\pi
\end{array} \right., \quad k \in \mathbb{Z}
\]

\[
\theta = \left\{ \begin{array}{l}
\frac{\pi}{8} + k\pi \\
\frac{3\pi}{8} + k\pi
\end{array} \right., \quad k \in \mathbb{Z}
\]
Solve the equation: $2\sin(2\theta) - \sqrt{2} = 0$, $0 \leq \theta < 2\pi$
solve(2 \sin(2^{\theta}) - \sqrt{2} = 0, 0 \leq \theta \leq 2\pi)

N.B. Book uses $0 \leq \theta < 2\pi$

ie. excludes endpoint $\theta = 2\pi$

\begin{align*}
\theta &= \frac{\pi}{8} \\
\theta &= \frac{3\pi}{8} \\
\theta &= \frac{9\pi}{8} \\
\theta &= \frac{11\pi}{8}
\end{align*}
solve\(2 \sin(\theta) - \sqrt{2} = 0\), \(\theta \geq 0\) and \(\theta \leq 2\pi\)

\[
solve(2 \cdot \sin(2 \cdot \theta) - \sqrt{2} = 0, \theta) \Rightarrow \theta = \frac{11\pi}{8} \text{ or } \theta = \frac{9\pi}{8} \text{ or } \theta:
\]

\[
\text{...} \quad \mathbf{f}(2) = 0, \theta \geq 0 \text{ and } \theta \leq 2\pi
\]
Solve the equation: \(2 \sin(2\theta) - \sqrt{2} = 0, 0 \leq \theta < 2\pi\)

\[
\sin(2\theta) = \frac{\sqrt{2}}{2}
\]

\[
2\theta = \begin{cases} 
\frac{\pi}{4} + k2\pi \\
\pi - \frac{\pi}{4} + k2\pi
\end{cases}, \quad k \in \mathbb{Z}
\]

\[
2\theta = \begin{cases} 
\frac{\pi}{4} + k2\pi \\
\frac{3\pi}{4} + k2\pi
\end{cases}
\]

\[
\theta = \begin{cases} 
\frac{\pi}{8} + k\pi \\
\frac{3\pi}{8} + k\pi
\end{cases}
\]
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \(2\sin(2\theta) - \sqrt{2} = 0, \ 0 \leq \theta < 2\pi\)

\[
\theta = \begin{cases} 
\frac{\pi}{8} + k\pi \\
\frac{3\pi}{8} + k\pi \\
\frac{\pi}{8}(1 + k8) \\
\frac{\pi}{8}(3 + k8)
\end{cases}
\]
Solve the equation: \( \sqrt{3} \tan(3\theta) + 1 = 0, \ 0 \leq \theta < 2\pi \)

\[
\tan(3\theta) = -\frac{1}{\sqrt{3}}
\]

\[
3\theta = -\frac{\pi}{6} + k\pi
\]

\[
\theta = -\frac{\pi}{18} + k\frac{\pi}{3} = \frac{\pi}{18} + k\frac{\pi}{18}
\]

\[
= \frac{\pi}{18} (-1 + k6)
\]
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: $\sqrt{3} \tan(3\theta) + 1 = 0, \ 0 \leq \theta < 2\pi$

$$\tan(3\theta) = -\frac{1}{\sqrt{3}}$$

$$3\theta = -\frac{\pi}{6} + n\pi, \ n \in \mathbb{Z}$$

$$\theta = -\frac{\pi}{18} + \frac{n\pi}{3}$$

$$= -\frac{\pi}{18}(1 - 6n)$$
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: $\sqrt{3} \tan(3\theta) + 1 = 0, \ 0 \leq \theta < 2\pi$

$$\theta = -\frac{\pi}{18}(1 - 6n)$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$-\pi/18$</td>
<td>$5\pi/18$</td>
<td>$11\pi/18$</td>
<td>$17\pi/18$</td>
<td>$23\pi/18$</td>
<td>$29\pi/18$</td>
<td>$35\pi/18$</td>
<td>$41\pi/18$</td>
</tr>
</tbody>
</table>

Graph showing $y = \sqrt{3} \tan(3\theta) + 1$ with marked points on the x-axis.
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \( \cos \left( \theta - \frac{\pi}{4} \right) = 1, \ 0 \leq \theta < 2\pi \)

\[
\cos \left( \theta - \frac{\pi}{4} \right) = 1
\]

\[
\theta - \frac{\pi}{4} = 0 + k2\pi \\
\theta = \frac{\pi}{4} + k2\pi
\]

\( k \in \mathbb{Z} \)
Solve the equation: \( \cos \left( \theta - \frac{\pi}{4} \right) = 1, \ 0 \leq \theta < 2\pi \)

\[ \theta - \frac{\pi}{4} = 0 + k2\pi, \ k \in \mathbb{Z} \land 0 \leq \theta < 2\pi \]

\[ \theta = \frac{\pi}{4} + k2\pi \]

\[ = \frac{\pi}{4} (1 + k8) \]

\[ = \left\{ \frac{\pi}{4} \right\} \]
§6.3 Trigonometric Equations

2. Solve Trigonometric Equations Using a Calculator.

Use a calculator to solve the equation: \( \cos(\theta) = 0.2, \ 0 \leq \theta < 2\pi \). Express any solutions in radians, rounded to two decimal places.
Solve the equation: \(2\cos^2(\theta) - \cos(\theta) - 1 = 0\), \(0 \leq \theta < 2\pi\).

Let: \(x = \cos(\theta)\)

\[0 = 2x^2 - x - 1 = 0\]
\[= (2x + 1)(x - 1)\]

\[2x + 1 = 0\]
\[x = -\frac{1}{2}\]

\[2 \cos(\theta) + 1 = 0\]
\[\cos(\theta) = -\frac{1}{2}\]

\[\theta = \frac{2\pi}{3} + k\pi\]
\[= \frac{4\pi}{3} + k\pi\]
\[\text{or} \quad \theta = -\frac{2\pi}{3} + k\pi\]
\[= \frac{-2\pi}{3} + k\pi\]

\[\cos(\theta) = 1\]

\[\theta = \theta + k2\pi\]
Solve the equation: \(2\cos^2(\theta) - \cos(\theta) - 1 = 0\), \(0 \leq \theta < 2\pi\).

\[0 = 2\cos^2(\theta) - \cos(\theta) - 1 \iff \text{Let } x = \cos(\theta)\]

\[= 2x^2 - x - 1\]

\[= (2x + 1)(x - 1)\]

Which requires:

\[2\cos(\theta) + 1 = 0 \implies \cos(\theta) = -\frac{1}{2}\]

\[\cos(\theta) - 1 = 0 \implies \cos(\theta) = 1\]
Solve the equation: \(\sin^2(\theta) - \sin(\theta) = \cos^2(\theta)\), \(0 \leq \theta < 2\pi\).

\[
\sin^2(\theta) - \sin(\theta) = \cos^2(\theta)
\]

\[
= 1 - \sin^2(\theta)
\]

\[
2\sin^2(\theta) - \sin(\theta) - 1 = 0 \iff \text{Let } x = \sin(\theta)
\]

\[
2x^2 - x - 1 =
\]

\[
(2x + 1)(x - 1) =
\]

Which requires:

\[
2\sin(\theta) + 1 = 0 \implies \sin(\theta) = -\frac{1}{2}
\]

\[
\sin(\theta) - 1 = 0 \implies \sin(\theta) = 1
\]
Which requires:

\[ 2 \sin(\theta) + 1 = 0 \quad \Rightarrow \quad \sin(\theta) = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{6} + k2\pi \]

\[ = \left\{ \begin{array}{ll}
\frac{\pi}{6} + k2\pi \left( \frac{\pi}{2} \right) \\
\frac{5\pi}{6} + k2\pi \left( \frac{\pi}{2} \right)
\end{array} \right. \]

\[ = \left\{ \begin{array}{ll}
\frac{-\pi}{6} + k2\pi \left( \frac{\pi}{2} \right) \\
\frac{7\pi}{6} + k2\pi \left( \frac{\pi}{2} \right)
\end{array} \right. \]

\[ = \left\{ \begin{array}{ll}
\frac{-\pi}{6} \left( 1 - 12k \right) \\
\frac{7\pi}{6} \left( 7 + 12k \right)
\end{array} \right. \]

\[ \sin(\theta) - 1 = 0 \quad \Rightarrow \quad \sin(\theta) = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{2} + k2\pi \]

For \( \theta \mid 0 \leq \theta < 2\pi \)

\[ \therefore \quad \theta = \left\{ \frac{-\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2} \right\} \]

\[ = \left\{ \frac{\pi}{6}, \frac{11\pi}{6}, \frac{7\pi}{6} \right\} \]
N.B. Computer does not understand $\sin_{12}(\theta)$
Input must be in "calculator notation" $\Rightarrow \sin(\theta)_{12}$!
5. Solve Trigonometric Equations Using a Graphing Utility

Solve the equation: \(3\cos(x) + x = 4\), \(0 \leq \theta < 2\pi\). Express the solution(s) rounded to two decimal places.

\[3\cos(x) + x = 4\]
\[3\cos(x) = 4 - x\]
\[\cos(x) = \frac{4 - x}{3}\]
solve(3\cos(x)+x=4,x)

Input interpretation:

\[
\text{solve } 3\cos(x) + x = 4 \text{ for } x
\]

Solution over the reals:

\[x \approx 4.53358\]

Plot:
\[ y = 3 \cos(x) + x \]

\[ y = 4 \]

\[ \_ \leq y \leq \_ \quad \text{step:} \_ \_ \]