§6.3 Trigonometric Equations

Objectives

2. Solve Trigonometric Equations Using a Calculator.
3. Solve Trigonometric Equations Quadratic in Form.
Given $f(x) = x^2$, $x \geq 0$

a) Find $f^{-1}(x) = \sqrt{x}$

b) Solve $f(x) = 4$

$\sqrt{x} = 4$

$x = \pm \sqrt{4}$

$= \pm 2$
Determine whether \( \theta = \pi/4 \) is a solution of the equation:

\[
2\sin(\theta) + \sqrt{2} = 0 = f(\theta)
\]

Is \( \theta = 5\pi/4 \) a solution?

1. \( \text{Check if } \pi/4 \text{ is a solution} \)

\[
f\left(\frac{\pi}{4}\right) = 2\sin\left(\frac{\pi}{4}\right) + \sqrt{2} = 2\cdot\frac{\sqrt{2}}{2} + \sqrt{2} = 0
\]

2. \( \text{Check if } \frac{5\pi}{4} \text{ is a solution} \)

\[
f\left(\frac{5\pi}{4}\right) = 2\sin\left(\frac{5\pi}{4}\right) + \sqrt{2} = 2\cdot\left(-\frac{\sqrt{2}}{2}\right) + \sqrt{2} = 0
\]

3. \( \text{Other solutions?} \)

\[
\theta = \left\{ \frac{5\pi}{4} + n\pi \right\} \quad \text{i.e. a solution to } f(\theta) = 0 \quad n \in \mathbb{Z}
\]
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Determine whether $\theta = \pi/4$ is a solution of the equation:

$$2\sin(\theta) + \sqrt{2} = 0$$

Is $\theta = 5\pi/4$ a solution?

Let: $f(\theta) = 2\sin(\theta) + \sqrt{2}$

$$f(\pi/4) = 2\sin(\pi/4) + \sqrt{2}$$

$$= 2\frac{\sqrt{2}}{2} + \sqrt{2}$$

$$\neq 0$$

$$f(5\pi/4) = 2\sin(5\pi/4) + \sqrt{2}$$

$$= 2\left(-\frac{\sqrt{2}}{2}\right) + \sqrt{2}$$

$$= 0$$
Solve the equation: \(2 \cos(\theta) - \sqrt{3} = 0\)

Give a general formula for all solutions. List eight of the solutions.

\[
\begin{align*}
2 \cos(\theta) &= \sqrt{3} \\
\cos(\theta) &= \frac{\sqrt{3}}{2} \\
\theta &= \left\{ \frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi, \right. \\
&= \left\{ \frac{\pi}{6} (1 + n12), \frac{5\pi}{6} (-1 + n12) \right. \\
\end{align*}
\]
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \(2 \cos(\theta) - \sqrt{3} = 0\)

Give a general formula for all solutions. List eight of the solutions.

**General formula for \(\theta\)**

\[
\theta = \frac{\pi}{6} (1 + n12) = \Theta_1(n)
\]

\[
\frac{\pi}{6} (-1 + n12) = \Theta_2(n)
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\Theta_1)</th>
<th>(\Theta_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\frac{\pi}{6})</td>
<td>(-\frac{\pi}{6})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{13\pi}{6})</td>
<td>(\frac{14\pi}{6})</td>
</tr>
<tr>
<td>-1</td>
<td>(-\frac{11\pi}{6})</td>
<td>(-\frac{13\pi}{6})</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{25\pi}{6})</td>
<td>(\frac{23\pi}{6})</td>
</tr>
</tbody>
</table>

Circle representation
- Solution set has an infinite # of elements
- Partial solution set (i.e. the list eight ... )

\[ \theta = \left\{ -\frac{\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}, 12\frac{\pi}{6}, -\frac{13\pi}{6}, -\frac{25\pi}{6}, -23\frac{\pi}{6} \right\} \]
Solve the equation: $2\cos(\theta) - \sqrt{3} = 0$

Give a general formula for all solutions. List eight of the solutions.

$$\theta = \pm \frac{\pi}{6} (1 + k12)$$
\[ \theta = \pi \left( 2n - \frac{1}{6} \right) \quad \text{and} \quad n \in \mathbb{Z} \]

\[ \theta = \pi \left( 2n + \frac{1}{6} \right) \quad \text{and} \quad n \in \mathbb{Z} \]

\( n \) is an element in the set of integers \( \mathbb{Z} = \{ 0, \pm 1, \pm 2, \ldots \} \).
\[ \theta = \frac{(12 \cdot n + 1) \cdot \pi}{6} \] or \[ \theta = \frac{(12 \cdot n - 1) \cdot \pi}{6} \]

\( n \in \mathbb{Z} \), calculator notation for integers
Solve the equation: \(2 \cos(\theta) - \sqrt{3} = 0\)

Give a general formula for all solutions. List eight of the solutions.

\[
2 \cos \theta - \sqrt{3} = 0
\]

\[
2 \cos \theta = \sqrt{3}
\]

\[
\cos \theta = \frac{\sqrt{3}}{2}
\]

\[
\theta = \begin{cases} 
\frac{\pi}{6} + k2\pi \\
-\frac{\pi}{6} + k2\pi
\end{cases}, \quad k \in \mathbb{Z}
\]

\[
\theta = \pm \frac{\pi}{6} + k2\pi
\]

\[
= \pm \frac{\pi}{6} (1 + k12)
\]
1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \(2 \cos(\theta) - \sqrt{3} = 0\)

Give a general formula for all solutions. List eight of the solutions.

\[
\theta = \left\{ \pm \frac{\pi}{6}, \pm \frac{11\pi}{6}, \pm \frac{13\pi}{6}, \pm \frac{23\pi}{6} \right\}
\]
1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \(2 \sin(2\theta) - \sqrt{2} = 0, \, 0 \leq \theta < 2\pi\)

\[2 \sin(2\theta) - \sqrt{2} = 0\]
\[2 \sin(2\theta) = \sqrt{2}\]
\[
\sin(2\theta) = \frac{\sqrt{2}}{2}
\]

\[
2\theta = \left\{ \begin{array}{ll}
\frac{\pi}{4} + n\frac{\pi}{2} \\
\frac{3\pi}{4} + n\frac{\pi}{2}
\end{array} \right\} \frac{1}{2}
\]

\[
\theta = \left\{ \begin{array}{ll}
\frac{\pi}{8} + n\frac{\pi}{2} \\
\frac{3\pi}{8} + n\frac{\pi}{2}
\end{array} \right\}, \quad n \in \mathbb{Z}
\]
Solve the equation: $2\sin(2\theta) - \sqrt{2} = 0, \; 0 \leq \theta < 2\pi$\

$\Rightarrow \sin(2\theta) = \sqrt{2}$

$\theta = \left\{ \begin{array}{l}
\frac{\pi}{8} + n\pi \\
\frac{3\pi}{8} + n\pi
\end{array} \right.$

\[= \left\{ \begin{array}{l}
\frac{\pi}{8} (1 + n\frac{8}{\pi}) \\
\frac{\pi}{8} (3 + n\frac{8}{\pi})
\end{array} \right. = \left\{ \begin{array}{l}
\frac{\pi}{8} (1 + n8) \\
\frac{\pi}{8} (3 + n8)
\end{array} \right. \]

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = \frac{\pi}{8} (1 + n8)$</td>
<td>$\frac{\pi}{8}$</td>
<td>$\frac{9\pi}{8}$</td>
<td>$\frac{17\pi}{8}$ = $\frac{16\pi}{8} + \frac{\pi}{8} &gt; 2\pi$</td>
</tr>
<tr>
<td>$\theta = \frac{\pi}{8} (3 + n8)$</td>
<td>$\frac{3\pi}{8}$</td>
<td>$\frac{11\pi}{8}$</td>
<td>$\frac{19\pi}{8}$ = $\frac{16\pi}{8} + \frac{3\pi}{8} &gt; 2\pi$</td>
</tr>
</tbody>
</table>

Because $0 \leq \theta < 2\pi$

Solution set is $\theta = \{ \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8} \}$
6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \(2 \sin(2\theta) - \sqrt{2} = 0, \ 0 \leq \theta < 2\pi\)

Solution set is \(\theta = \left\{ \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8} \right\}\)
\[ \text{solve}(2 \sin(2 \theta) - \sqrt{2} = 0, 0 \leq \theta < 2\pi) \]

Results:
\[ \theta = \frac{\pi}{8} \]
\[ \theta = \frac{3\pi}{8} \]
\[ \theta = \frac{9\pi}{8} \]
\[ \theta = \frac{11\pi}{8} \]
solve\((2 \sin(\theta) - \sqrt{2}) = 0\), \(\theta \geq 0 \text{ and } \theta \leq 2\pi\)

for newer TI models, OK to use:

\(0 \leq \theta < 2\pi\)

instead of

\(|\theta| = 0 \text{ and } \theta < 2\pi|
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: \(2 \sin(2\theta) - \sqrt{2} = 0, \ 0 \leq \theta < 2\pi\)

\[
\sin(2\theta) = \frac{\sqrt{2}}{2}
\]

\[
2\theta = \begin{cases}
\frac{\pi}{4} + k2\pi \\
\frac{\pi}{4} - \frac{\pi}{4} + k2\pi
\end{cases}, \ k \in \mathbb{Z}
\]

\[
2\theta = \begin{cases}
\frac{\pi}{4} + k2\pi \\
\frac{3\pi}{4} + k2\pi
\end{cases}
\]

\[
\theta = \begin{cases}
\frac{\pi}{8} + k\pi \\
\frac{3\pi}{8} + k\pi
\end{cases}
\]
Solve the equation: \( 2 \sin(2\theta) - \sqrt{2} = 0, \ 0 \leq \theta < 2\pi \)

\[
\theta = \begin{cases} 
\frac{\pi}{8} + k\pi \\
\frac{3\pi}{8} + k\pi \\
\frac{\pi}{8}(1 + k8) \\
\frac{\pi}{8}(3 + k8)
\end{cases}
\]
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: $\sqrt{3} \tan(3\theta) + 1 = 0, \ 0 \leq \theta < 2\pi$

\[
\tan(\alpha) = -\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha \text{ is angle whose tangent is } -\frac{1}{\sqrt{3}}
\]

\[
\alpha = -\frac{\pi}{6} + n\pi \quad \Rightarrow \text{N.B. } n\pi \text{ not } n2\pi \text{ because period of tangent function is } \pi
\]

\[
3\theta = -\frac{\pi}{6} + n\pi
\]

\[
\theta = -\frac{\pi}{18} + n\frac{\pi}{3} \left(\frac{1}{6} \cdot \frac{6}{1}\right)
\]

\[
= -\frac{\pi}{18} \left(1 - n6\right)
\]
1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: $\sqrt{3} \tan (3\theta) + 1 = 0, \ 0 \leq \theta < 2\pi$

\[
\tan (3\theta) = -\frac{1}{\sqrt{3}} \\
3\theta = -\frac{\pi}{6} + n\pi, \ n \in \mathbb{Z}
\]

\[
\theta = -\frac{\pi}{18} + \frac{n\pi}{3} \left(\frac{1}{6}, \frac{6}{1}\right) \\
= -\frac{\pi}{18} (1 - 6n)
\]
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: $\sqrt{3} \tan(3\theta) + 1 = 0$, $0 \leq \theta < 2\pi$

$\theta = -\frac{\pi}{18} (1 - 6n)$

<table>
<thead>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$-\pi/18$</td>
<td>$5\pi/18$</td>
<td>$11\pi/18$</td>
<td>$17\pi/18$</td>
<td>$23\pi/18$</td>
<td>$29\pi/18$</td>
<td>$35\pi/18$</td>
<td>$41\pi/18$</td>
</tr>
</tbody>
</table>
§6.3 Trigonometric Equations

1. Solve Equations Involving a Single Trigonometric Function

Solve the equation: $\cos \left( \theta - \frac{\pi}{4} \right) = 1$, $0 \leq \theta < 2\pi$

\[
\cos \left( \theta - \frac{\pi}{4} \right) = 1
\]

\[
\theta - \frac{\pi}{4} = 0 + n \cdot 2\pi
\]

\[
\theta = 0 + n \cdot 2\pi + \frac{\pi}{4} = \frac{\pi}{4} (1 + n \cdot \frac{4}{\pi}) = \frac{\pi}{4} (1 + 8n)
\]

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</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{3\pi}{4}$</td>
</tr>
</tbody>
</table>

$\frac{3\pi}{4} = \frac{8\pi}{4} + \frac{\pi}{4} > 2\pi$

So solution set $\theta = \left\{ \frac{\pi}{4} \right\}$
Solve the equation: $\cos\left(\theta - \frac{\pi}{4}\right) = 1, \ 0 \leq \theta < 2\pi$

$\theta - \frac{\pi}{4} = 0 + k2\pi, \ k \in \mathbb{Z} \land 0 \leq \theta < 2\pi$

$\theta = \frac{\pi}{4} + k2\pi$

$= \frac{\pi}{4} (1 + k8)$

$= \left\{ \frac{\pi}{4} \right\}$
2. Solve Trigonometric Equations Using a Calculator.

Use a calculator to solve the equation: \( \cos(\theta) = 0.2, \ 0 \leq \theta < 2\pi \).
Express any solutions in radians, rounded to two decimal places.
\[ y = \cos(\theta) \]

So \( \theta \approx 1.37, 4.91 \)
Solve the equation: $2\cos^2(\theta) - \cos(\theta) - 1 = 0$, $0 \leq \theta < 2\pi$.

Let $x = \cos(\theta)$

$2x^2 - x - 1 = 0$

$(2x + 1)(x - 1) = 0$

$2x + 1 = 0$

$x = -\frac{1}{2}$

$\cos(\theta) = -\frac{1}{2}$

$\theta = \left\{ \frac{\pi}{3} + n \cdot 2\pi, \quad -\frac{2\pi}{3} + n \cdot 2\pi \right\}$
\[ \theta = \sum \left\{ \begin{array}{ll}
\frac{2\pi}{3} + n2\pi \\
-\frac{2\pi}{3} + n2\pi
\end{array} \right. \\
= \sum \left\{ \begin{array}{ll}
\frac{2\pi}{3} (1 + n2\pi - \frac{3}{2\pi}) \\
\frac{2\pi}{3} (-1 + n3)
\end{array} \right. 
\]
<table>
<thead>
<tr>
<th>η</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2\pi}{3}(1+\eta^3)$</td>
<td>$\frac{2\pi}{3}$</td>
<td>$\frac{8\pi}{3} = \frac{6\pi}{3} + \frac{2\pi}{3} &gt; 2\pi$</td>
<td></td>
</tr>
<tr>
<td>$\frac{2\pi}{3}(-1+\eta^3)$</td>
<td>$-\frac{2\pi}{3} &lt; 0$</td>
<td>$\frac{4\pi}{3}$</td>
<td>$\frac{14\pi}{3} &gt; 2\pi$</td>
</tr>
</tbody>
</table>

\[ \therefore \theta = \left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\} \]
Solve the equation: \(2\cos^2(\theta) - \cos(\theta) - 1 = 0, 0 \leq \theta < 2\pi\).

\[
0 = 2\cos^2(\theta) - \cos(\theta) - 1 \iff \text{Let } x = \cos(\theta)
\]

\[
= 2x^2 - x - 1
\]

\[
= (2x + 1)(x - 1)
\]

Which requires:

\[
2\cos(\theta) + 1 = 0 \implies \cos(\theta) = -\frac{1}{2}
\]

\[
\cos(\theta) - 1 = 0 \implies \cos(\theta) = 1
\]

\[
\therefore \theta = \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{3}
\]
\[ \sin^2(\theta) - \sin(\theta) = \cos^2(\theta), \quad 0 \leq \theta < 2\pi. \]

\[ \sin^2(\theta) - \sin(\theta) = \cos^2(\theta) \]
\[ = 1 - \sin^2(\theta) \]

\[ 2\sin^2(\theta) - \sin(\theta) - 1 = 0 \quad \Longleftrightarrow \text{Let } x = \sin(\theta) \]
\[ 2x^2 - x - 1 = \]
\[ (2x + 1)(x - 1) = \]

Which requires:
\[ 2\sin(\theta) + 1 = 0 \quad \Rightarrow \quad \sin(\theta) = -\frac{1}{2} \]
\[ \sin(\theta) - 1 = 0 \quad \Rightarrow \quad \sin(\theta) = 1 \]
§6.3 Trigonometric Equations

4. Solve Trigonometric Equations Using Fundamental Identities

Solve the equation: \( \sin^2(\theta) + \cos(\theta) = 3 \), \( 0 \leq \theta < 2\pi \).

\[
0 = 3 - \sin^2(\theta) - \cos(\theta)
\]

\[
= 3 - (1 - \cos^2(\theta)) - \cos(\theta)
\]

\[
= \cos^2(\theta) - \cos(\theta) + 2 \iff \text{Let } x = \cos(\theta)
\]

\[
x^2 - x + 2 \implies x = \frac{1}{2} \pm \sqrt{\frac{1}{2} - \frac{3}{2}} \iff \text{No real solution!}
\]

\[
x = \frac{1}{2} \pm i \sqrt{\frac{3}{2}} \quad \text{where } i = \sqrt{-1}
\]

N.B.: \( 0 \leq \sin^2(\theta) \leq 1 \) and \( -1 \leq \cos(\theta) \leq 1 \)
Solve the equation: $3\cos(x) + x = 4$, $0 \leq \theta < 2\pi$. Express the solution(s) rounded to two decimal places.
solve(3 \cos(x) + x = 4, x)

Input interpretation:

solve 3 \cos(x) + x = 4 for x

Solution over the reals:

x = 4.53358

Plot:
\[ y = 3 \cos(x) + x \]

\[ y = 4 \]

\[ 0 \leq y \leq 4 \quad \text{step: } \]

\((4.534, 4)\)